

Weak Values and Continuous-Variable Entanglement Concentration

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We demonstrate a general weak measurement model which allows Gaussian preserving entanglement concentration of the two mode squeezed vacuum. The power of this simple and elegant protocol is through the constraints it places on possible ancilla states and measurement strategies that will allow entanglement concentration. In particular, it is shown how previously discovered protocols of this kind emerge as special examples of the general model described here. Finally, as evidence of its utility, we use it to provide another novel example of such a protocol.

I. INTRODUCTION

As any textbook on quantum mechanics will testify, the measurable values of any observable \mathcal{O} coincide with the eigenvalues of an appropriate self-adjoint operator \hat{O} . However, in [1], *Aharonov* and *Vaidmann* showed that if a system with observable \mathcal{O} is both pre and post-selected in the states $|\Phi_1\rangle$ and $|\Phi_2\rangle$, then the observable can possess so-called weak values:

$$O_W = \frac{\langle \Phi_1 | \hat{O} | \Phi_2 \rangle}{\langle \Phi_1 | \Phi_2 \rangle}. \quad (1)$$

It is vital to note that (1) does not coincide with the eigenvalues of the operator \hat{O} and can in general be complex. One can transfer the weak values belonging to a particular system onto another via indirect quantum measurement as introduced by *Von Neumann* [2], with a caveat - the coupling strength between signal and probe must be weak [1]. Such models are collectively, rather appropriately, known as *weak measurements*. Despite initial controversy, the concept of weak values and measurements have enjoyed a surge of theoretical development [1, 3, 4, 5, 6], and eventual experimental confirmation [7].

Perhaps unexpectedly, weak measurements have an application in the currently open problem of entanglement distillation of Gaussian states. So far, one possible approach was provided by distillation protocols involving the subtraction of a definite number of photons [8, 9]. In contrast, the model reported here provides a general method for *Procrustean* entanglement distillation protocols which subtract or add an *indefinite* number of photons [10, 11]. These latter protocols probabilistically modify the average number of photons of a particular entangled Gaussian state whilst preserving its essential Gaussian features. Distillation of entangled resources is a major issue in quantum information processing, in particular for the development of quantum repeaters that allow for fault tolerant communication between different information processors. In this paper we address these tasks

using mesoscopic, continuous variables, a viable and extremely promising alternative to the traditional quantum bit-based information processing.

We have discovered that [10, 11] are special cases of a general weak measurement interaction. Using the weak value paradigm, we demonstrate how to construct a general model of such *Procrustean* protocols. Moreover, we identify that the features of these protocols, namely success conditions and Gaussian preservation are not unique to the particular choices advocated in both [10, 11]. Instead, our general analysis reveals that the origin of these features lie with the consequences of performing a weak measurement. Furthermore, our model constrains the pre and post-selected ancilla states whilst providing a method for determining which possible combinations work. Indeed, [10, 11] emerge as examples of a general rule and we provide another example which of a possible configuration.

The protocols [10, 11] were suggested as possible resolutions to the following quantum communication scenario. Suppose Alice and Bob wish to perform a particular continuous-variable entanglement assisted protocol. Further assume that they share a two-mode squeezed vacuum (TMSS) encoded in two light modes, as an entanglement resource [12, 13]:

$$|\zeta(\lambda)\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n, n\rangle. \quad (2)$$

To ensure maximum performance from their protocol, they must possess a high quality entangled state. Ergo, they wish to increase the entanglement of their shared entangled state before consuming it. Moreover, they require this to be done in a manner that will preserve its Gaussian features. Accordingly, they want to probabilistically map their initial TMSS $|\zeta(\lambda)\rangle$ to another more entangled one $|\zeta(\lambda')\rangle$.

II. THE PROTOCOL

To solve this problem we use the *Procrustean* method [14], where we probabilistically modify the *Schmidt* coefficients of the input state to generate an output state with a greater degree of entanglement whilst preserving the basis. The following configuration is advocated, as

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depicted in fig(1), the entangled state in modes A and B is coupled to ancilla state in mode C by means of a unitary evolution between B and C . The requirements of the *Procrustean* method dictate that the interaction Hamiltonian describing this process must be of the form

$$\hat{H}_I = \hbar\kappa(t)\hat{n}_B \otimes \hat{O}_C. \quad (3)$$

This is required to preserve the *Schmidt* basis of the TMSS, i.e. the *Fock* basis. Assuming the interaction persists for T seconds, then the corresponding unitary evolution operator is

$$\hat{U} = e^{-i \int_0^T \kappa(t) \hat{n}_B \hat{O}_C} = e^{-i\kappa_T \hat{n}_B \hat{O}_C}, \quad (4)$$

where $\kappa_T = \kappa(T) - \kappa(0)$.

Following this, Bob performs a measurement on the ancilla and post-selects it in the state $|\Phi_2\rangle$. Consequently, the state shared between Alice and Bob is given by

$$\begin{aligned} |\Psi_f\rangle &= \mathcal{N} \langle \Phi_2 | e^{-i\kappa_T \hat{n}_B \hat{O}_C} | \zeta(\lambda) \rangle | \Phi_1 \rangle \\ &= \mathcal{N}' \sum_{m=0}^{\infty} \frac{(-i\kappa_T)^m}{m!} \frac{\langle \Phi_2 | \hat{O}_C^m | \Phi_1 \rangle}{\langle \Phi_2 | \Phi_1 \rangle} \hat{n}_B^m | \zeta(\lambda) \rangle. \end{aligned} \quad (5)$$

The weak value of \hat{O}_C is defined as

$$O_W = \frac{\langle \Phi_2 | \hat{O}_C | \Phi_1 \rangle}{\langle \Phi_2 | \Phi_1 \rangle}, \quad (6)$$

and so the final state of the system is given as

$$|\Psi_f\rangle = \mathcal{N}' \exp(-i\kappa_T O_W \hat{n}_B) | \zeta(\lambda) \rangle, \quad (7)$$

if the weakness condition

$$\sum_{m=2}^{\infty} \frac{(-i\kappa_T)^m}{m!} \{O_W^m - (O_W)^m\} \hat{n}_B^m | \zeta(\lambda) \rangle \approx 0 | \phi \rangle \quad (8)$$

is obeyed. Here $|\phi\rangle$ is an arbitrary vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ and $O_W^m = \langle \Phi_2 | \hat{O}_C^m | \Phi_1 \rangle / \langle \Phi_2 | \Phi_1 \rangle$. By using the linear independence of the *Schmidt* basis of the TMSS we can express (8) as set of equations:

$$\lambda^n \left(\frac{\langle \Phi_2 | e^{-i\kappa_T n \hat{O}_C} | \Phi_1 \rangle}{\langle \Phi_2 | \Phi_1 \rangle} - e^{-i\kappa_T n O_W} \right) \approx 0 \quad \forall n \in [0, \infty). \quad (9)$$

Assuming that the above weakness condition is satisfied means that the output state is another TMSS as (7) yields

$$|\Psi_f\rangle = \sqrt{1 - \lambda^2 e^{2\kappa_T \text{Im}(O_W)}} \sum_{n=0}^{\infty} \lambda^n e^{-i\kappa_T O_W n} |n, n\rangle, \quad (10)$$

This only holds subject to $\lambda^2 e^{2\text{Im}(O_W)\kappa_T} < 1$, otherwise the output state is un-physical as the normalisation constant will not converge. From (10) it can be seen that the real part O_W induces a phase shift on the TMSS

whereas the imaginary part modifies the average number of photons in the state. Put succinctly, the induced transformation is $\lambda \rightarrow \lambda' = \lambda e^{-i\kappa_T O_W}$. Thus, the average number of photons has been altered [12],

$$\frac{2\lambda^2}{1 - \lambda^2} \rightarrow \frac{2\lambda^2 e^{2\kappa_T \text{Im}(O_W)}}{1 - \lambda^2 e^{2\kappa_T \text{Im}(O_W)}}, \quad (11)$$

and as a result the entanglement content of the state is also modified. It is in this sense that we can subtract or add an indefinite number of photons to our target state. To verify entanglement concentration, we use method of majorization. Let $\mathbf{d} = (d_0^2, d_1^2, \dots)^T$ be the ordered vector of the eigenvalues of the reduced density matrices of (7) and $\mathbf{c} = (c_0^2, c_1^2, \dots)^T$ be the analogous object for the initial TMSS. Then (7) is more entangled than $|\zeta(\lambda)\rangle$ if its reduced density matrices are more mixed. This will be the case if \mathbf{d} is *majorized* by \mathbf{c} , which is written as $\mathbf{d} \prec \mathbf{c}$ and defined by [15, 16]

$$\sum_{k=\ell}^{\infty} d_k^2 \geq \sum_{k=\ell}^{\infty} c_k^2, \quad (12)$$

for $\ell \in [1, \infty)$. This follows since measures of bipartite pure state entanglement such as the *Von Neumann* entropy and the purity belong to the *Shur concave* [16] and hence, preserve the majorization order

$$\mathbf{d} \prec \mathbf{c} \implies f(\mathbf{d}) \geq f(\mathbf{c}). \quad (13)$$

It is sufficient for entanglement concentration to show that the eigenvalues of the reduced density matrices of the output state majorize those of the input state.

Thus, applying the majorization condition to the states (7) and the initial TMSS yields

$$\left(\lambda e^{2\kappa_T \text{Im}(O_W)} \right)^\ell > \lambda^\ell, \quad (14)$$

$\forall \ell \in [1, \infty]$. The only way to satisfy (14) is if the imaginary part of O_W is positive for all ℓ (assuming $\kappa_T > 0$).

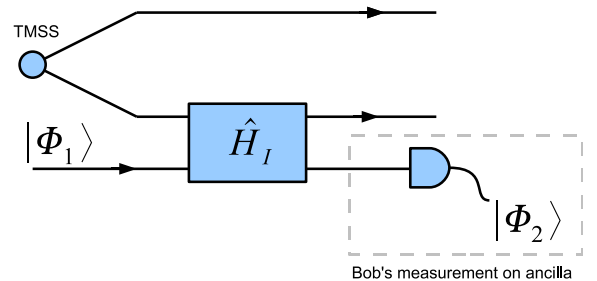


Figure 1: Bob mixes his half of the TMSS with an ancillary mode pre-selected in $|\Phi_1\rangle$ via a non-linear photonic interaction described by the Hamiltonian \hat{H}_I . The ancilla mode is then subjected to a post-selected measurement leaving it in the state $|\Phi_2\rangle$.

Entanglement concentration then only occurs if

$$\text{Im}g\left(\frac{\langle\Phi_2|\hat{O}_C|\Phi_1\rangle}{\langle\Phi_2|\Phi_1\rangle}\right) > 0. \quad (15)$$

Consequently, the protocol provides a success condition for entanglement concentration based on the properties of the weak value of a particular observable. This in conjunction with condition (9) provides a number of constraints that the interaction Hamiltonian \hat{H}_I , the pre-selected and post-selected ancilla states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ and the observable \hat{O}_C must obey in order to produce entanglement concentration of the TMSS. It is interesting to note that the weak condition (9) coupled with the requirements of the *Procrustean* method are all that is required to preserve the gaussian character of the TMSS.

III. EXAMPLES

We now demonstrate that previously discovered protocols of this type can emerge as special examples of the general model advocated here. We will also calculate the associate weak values and demonstrate that the weakness condition is satisfied. The previous schemes [10] and [11], required that Bob's half of the TMSS be mixed with an ancillary coherent state $|\alpha\rangle$, where $\alpha \in \mathbb{R}$ and $\alpha > 0$, in a non-linear medium exhibiting the cross Kerr effect $\hat{H}_I = \hbar\kappa(t)\hat{n}_B\hat{n}_C$ before being subjected to a measurement and post-selection condition. Using the success condition (15), we can derive a constraint on the possible post-selected ancilla states which will allow us to select measurement strategies the lead to Gaussian-preserving entanglement concentration. Thus, we are interested in the weak values of the number operator \hat{n}_C :

$$n_W = \frac{\langle\Phi_2|\hat{n}_C|\alpha\rangle}{\langle\Phi_2|\alpha\rangle} = \frac{e^{-\alpha^2/2}\alpha\partial_\alpha\left(e^{\alpha^2/2}\langle\Phi_2|\alpha\rangle\right)}{\langle\Phi_2|\alpha\rangle}. \quad (16)$$

The second equality in (16) follows from $\alpha\partial_\alpha(\alpha^n) = n\alpha^n$. Furthermore, if we assume

$$\langle\Phi_2|\alpha\rangle = R(\alpha)e^{i\theta(\alpha)}, \quad (17)$$

where $R(\alpha)$ and $\theta(\alpha)$ are the magnitude and phase of the scalar product of $\langle\Phi_2|\alpha\rangle$, then after some algebra (16) can be written as

$$n_W = \alpha^2 + \frac{\alpha}{R(\alpha)}\frac{\partial R}{\partial\alpha} + i\alpha\frac{\partial\theta}{\partial\alpha}. \quad (18)$$

Consequently, the success condition requires that $\text{Im}g(n_W) > 0 \Leftrightarrow \alpha\partial_\alpha\theta(\alpha) > 0$. Thus, the only variants of this family of protocols which achieve the desired effect are those where the phase of $\langle\Phi_2|\alpha\rangle$ is a monotonic increasing function of α . This prediction allows us to recover previously suggested protocols and uncover new variants.

Fiurášek, Mišta and Filip (2003) [10]. In this scheme, the ancillary coherent state is projected onto $|\beta\rangle = ||\beta|e^{i\phi}\rangle$ via eight-port-Homodyne detection. This example prevails due to the over-complete nature of coherent states

$$\langle\beta|\alpha\rangle = e^{-\alpha^2/2}e^{-|\beta|^2/2}e^{\alpha\beta^*}, \quad (19)$$

where it is clear that the phase of the above is a monotonic increasing function of α only if the imaginary part of β is negative. This also follows from

$$\text{Im}g(n_W) = \alpha\partial_\alpha\theta(\alpha) = -\alpha|\beta|\sin\phi. \quad (20)$$

Hence, the success condition for this protocol is given by $\pi < \phi < 2\pi$ and only states post-selected with respect to this condition will allow the desired effect. The weakness condition is then given as

$$\lambda^n \left(\frac{\langle\beta|e^{-i\kappa_T n \hat{n}_C}|\alpha\rangle}{\langle\beta|\alpha\rangle} - e^{-i\kappa_T n \alpha\beta^*} \right) = 0 \quad \forall n \in [0, \infty), \quad (21)$$

Using the identity $\exp(\sigma\hat{a}^\dagger\hat{a}) =: \exp(\{e^\sigma - 1\}\hat{a}^\dagger\hat{a})$: [12], then (21) becomes

$$\lambda^n \left(e^{(e^{-i\kappa_T n} - 1)\beta^* \alpha} - e^{-i\kappa_T n \beta^* \alpha} \right) = 0 \quad \forall n \in [0, \infty). \quad (22)$$

The above is true if $\kappa_T \ll 1$ such that $e^{-i\kappa_T n} \approx 1 - i\kappa_T n$, which only holds for sufficiently small n . Thus, for small values of n , (22) is satisfied. However, for large values of n where $e^{-i\kappa_T n} \neq 1 - i\kappa_T n$, (22) still holds because $\lambda < 1$ and hence $\lambda^n \rightarrow 0$ for progressively larger n . Thus, the weakness condition requires a balancing act between the non-linear coupling and the squeezing of the input TMSS. The authors of [10] arrive at the same conclusion.

Menzies and Korolkova (2006) [11]. Here balanced Homodyne detection is employed by Bob, in other words, the post-selected state of the ancilla is the quadrature eigenstate $|x_\phi\rangle = |\Phi_2\rangle$ where $\hat{X}_\phi|x_\phi\rangle = x_\phi|x_\phi\rangle$ and $\hat{X}_\phi = 2^{-1/2}(e^{i\phi}\hat{a}^\dagger + e^{-i\phi}\hat{a})$. Once again, this protocol works because of the nature of the overlap between the pre- and post-selected states. In this case, we have [12]

$$\langle x_\phi|\alpha\rangle = \pi^{-1/4}e^{-x_\phi^2/2 + \sqrt{2}e^{-i\phi}x_\phi\alpha - e^{-2i\phi}\alpha^2/2}, \quad (23)$$

then the imaginary part of the weak value is

$$\text{Im}g(n_W) = \alpha\partial_\alpha\theta = \sqrt{2}\alpha\sin\phi x_\phi - \alpha^2\sin(2\phi), \quad (24)$$

and so condition (15) translates to

$$\text{Im}g(n_W) > 0 \Leftrightarrow x_\phi > \sqrt{2}\alpha\cos\phi. \quad (25)$$

Every possible quadrature measurement has its own success condition for entanglement concentration. This condition defines the post-selection criterion. The weakness condition for this protocol is given as

$$\lambda^n \left(\frac{\langle x_\phi|e^{-i\kappa_T n \hat{n}_C}|\alpha\rangle}{\langle x_\phi|\alpha\rangle} - e^{-i\kappa_T n n_W} \right) = 0, \quad \forall n \in [0, \infty). \quad (26)$$

This can be re-expressed as ($\forall n \in [0, \infty)$):

$$\lambda^n \left[\exp \left(\sqrt{2} x_\phi \alpha e^{i\phi - i\kappa_T n} - \alpha^2 \frac{e^{2i\phi - 2i\kappa_T n}}{2} \right) - \exp \left(-i\kappa_T n \left\{ \sqrt{2} x_\phi \alpha e^{i\phi} - \alpha^2 e^{2i\phi} \right\} \right) \right] = 0.$$

So, just as for the previous example, we see that (26) is equivalent to (22). Thus, both schemes require the balancing between the initial *Schmidt* coefficients and the magnitude of the non-linear coupling.

Squeezed vacuum post-selection scheme. To generate further examples, we simply need to identify further quantum optical states that satisfy $\partial\alpha\theta(\alpha) > 0$. An immediate and obvious choice is given by selecting the post-selected state as a single mode squeezed vacuum $|\Phi_2\rangle = |re^{i\phi}\rangle$ since [12]

$$\langle re^{i\phi} | \alpha \rangle = \sqrt{\text{sech } r} \exp \left(-\frac{\alpha^2}{2} \{1 + e^{-i\phi} \tanh r\} \right). \quad (27)$$

Consequently, the phase of the above overlap is given as

$$\theta(\alpha) = \frac{\alpha^2}{2} \sin \phi \tanh r, \quad (28)$$

and hence, in this example, the success condition (15) is

$$\text{Im}(n_W) = \alpha^2 \tanh r \sin \phi > 0 \Leftrightarrow 0 < \phi < \pi/2. \quad (29)$$

The weak condition is expressed as

$$\lambda^n \left(\frac{\langle re^{i\phi} | e^{-i\kappa_T n \hat{n}_C} | \alpha \rangle}{\langle re^{i\phi} | \alpha \rangle} - e^{-i\kappa_T n n_W} \right) = 0, \quad \forall n \in [0, \infty), \quad (30)$$

where the first term on the LHS is $\exp \left(-\frac{\alpha^2}{2} \{e^{2in\kappa_T} - 1\} e^{-i\phi} \tanh r \right)$ and the second is $\exp(-i\kappa_T n \alpha^2 e^{-i\phi} \tanh r)$. Clearly (30) can only be satisfied if $\kappa_T n \ll 1$. Note that for large n , (30) holds because $\lambda^n \rightarrow 1$. In this example, the power of the weak value approach is evident due to its ability to provide an elegant shortcut to the required operation conditions.

IV. CONCLUDING REMARKS

In conclusion, we have illustrated an application of weak measurements for a Gaussian preserving entanglement concentration. In particular, we have provided

a general weak measurement model that allows for the probabilistic transformation of an input TMSS $|\zeta(\lambda)\rangle$ to an output $|\zeta(\lambda')\rangle$. This model allows several useful features. Firstly, we can formulate an entanglement concentration success condition dependant on the magnitude of the imaginary part of the weak value. Moreover, this success condition also puts a constrain on the initial ancilla states and the subsequent measurement strategy employed. Secondly, the weakness criterion in conjunction with the requirements of the *Procrustean* method guarantee that the entanglement concentration will be Gaussian preserving. As a consequence of the universality of the model provided here, it is a simple matter to determine novel examples of this protocol by appealing to the imaginary part of the ancilla's weak value. Indeed, we attempt to justify this viewpoint by providing another example of this entanglement concentration protocol inspired purely from observations of the imaginary weak value.

Despite the advances offered by adopting the weak measurement paradigm here, there remain a number of outstanding problems. For example, can we generalize this model to account for type-preserving *Procrustean* entanglement concentration for arbitrary continuous-variable pure bipartite entangled states? Furthermore, we note that the weak condition itself is never totally satisfied. Indeed, whereas it is easy to satisfy (9) for small and large n by requiring that $\lambda < 1$ and $\kappa_T \ll 1$, it is, however, not clear if weakness condition is true for intermediate values of n . It is then natural to enquire the consequences of, if any, such violations to the ability of the protocol to deliver Gaussian preserving entanglement concentration. In principle, we would like to obtain a quantitative relation between the magnitude of the violation and the ability of (15) to act as a success condition. The weakness condition allows the Gaussian preservation and so failure of the condition results in a non-Gaussian output state. Thus, we could measure the violation of (9) by measuring the extent of the non-Gaussian character of the output state. However, obtaining such a quantitative relation is highly non-trivial as the features of the non-Gaussian entangled states cannot, in many cases, be calculated in an analytical fashion. Consequently, the formulation of such a relation remains a goal for future research.

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